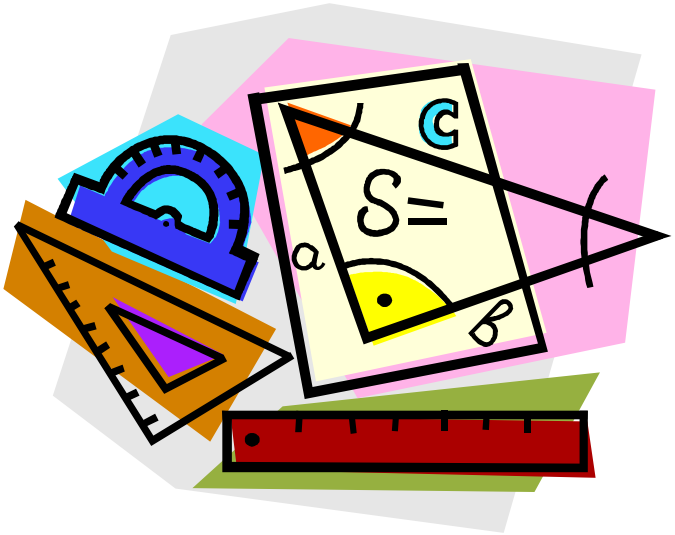
Comprehensive



Do NOT open until you are told to do so.

March 22, 2018

1. During a recent police investigation, Chief Inspector Stone was interviewing five local villains to try and identify who stole Mrs. Archer's cake from the fair. Below is a summary of their statements:

   Arnold: “It wasn’t Edward.” “It was Brian.”

   Brian: “It wasn’t Charles.” “It wasn’t Edward.”

   Charles: “It was Edward.” “It wasn’t Arnold.”

   Derek: “It was Charles.” “It was Brian.”

   Edward: “It was Derek.” “It wasn’t Arnold”

   It was well known that each suspect told exactly one lie. Which suspect did it?

   a. Edward b. Derek c. Arnold d. Brian e. Charles

2. What is the domain of $f(x)=2(\log(x-1))-\frac{1}{\sqrt{4-x^2}}$

   a. $(-2,2)$ b. $(1,2)$ c. $(1,\infty)$ d. $(-2,1)\cup(1,2)$ e. $(-2,1)$

3. If $\sin(x)+\cos(x)=\frac{1}{4}$, then what is the value of $\sin^3(x)+\cos^3(x)$?

   a. $\frac{5}{26}$ b. $\frac{11}{32}$ c. $\frac{31}{64}$ d. $\frac{47}{128}$ e. $\frac{59}{256}$

4. The angles of a triangle are all prime numbers. What is the measure of the largest possible angle?

   a. 169° b. 171° c. 173° d. 177° e. 179°

5. To increase the area of a circle by 44%, by what percentage must you increase the radius?

   a. 20% b. 22% c. 25% d. 22.5% e. 18%

6. The line $y = mx + b$ is tangent to the circle $(x+1)^2 + (y-1)^2 = 25$ at $(3,4)$. What is $m+b$?

a. $\frac{5}{12}$ b. $\frac{5}{2}$ c. $\frac{7}{2}$ d. $\frac{20}{3}$ e. $\frac{35}{4}$

7. The area of the four-sided region in the first quadrant bounded by the *x*-axis, *y*-axis, and the lines $3x+4y=12$ and $2y-x=2$ is cut in half by the line $y=kx$. What is k?

a. $\frac{33}{76}$ b. $\frac{2}{5}$ c. $\frac{11}{19}$ d. $\frac{1}{2}$ e. $\frac{21}{38}$

8. Suppose that $f(x) = \frac{x^3 + x^2 + cx + d}{x+2}$ is equivalent to $g(x) = ax^2 + bx + 4$ on its domain. What is $f(3)$?

a. $\frac{38}{5}$ b. 9 c. 10 d. $\frac{54}{5}$ e. 12

9. Compute the following: $\sum_{n=1}^{2018} \left(i^n + i^{-n}\right)$

a. 2 b. −2 c. 0 d. $i-2$ e. $2-i$

10. Which of the following is equivalent to $\frac{\tan(\theta)}{1-\cos(2\theta)}$?

a. $2\sec(\theta)\csc(\theta)$ b. $\frac{2\cot(\theta)}{\sec(\theta)}$ c. $\frac{\cos(\theta)}{2\sin(\theta)}$ d. $\frac{\sin(\theta)}{2\cos(\theta)}$ e. $\frac{1}{2}\csc(\theta)\sec(\theta)$

11. The numbers p, q, r, s, and t are consecutive positive integers, arranged in increasing order. If $p+q+r+s+t$ is a perfect cube and $q+r+s$ is a perfect square, then what is the smallest possible value of r?

a. 75 b. 288 c. 225 d. 675 e. 725

12. One of the Roman dice in the British Museum has 6 square faces and 8 triangular faces. It is twice as likely to land on any given square face as any given triangular face. What are the odds of landing on a triangular face if this particular die is thrown?

a. $\frac{2}{5}$ b. $\frac{3}{4}$ c. $\frac{4}{3}$ d. $\frac{3}{2}$ e. $\frac{2}{3}$

13. Assume the measures of the three angles of a triangle in degrees are integers. How many of these triangles exist such that the measures of all three angles are perfect squares?

a. 0 b. 1 c. 2 d. 3 e. 4

14. A cube measuring 6 inches on an edge is painted red. The cube is then cut into cubes that measure 3 inches on an edge and the unpainted faces are painted green. Finally, these cubes are cut into unit cubes and their unpainted faces are painted blue. How many faces are painted blue on the unit cubes?

a. 864 b. 756 c. 648 d. 432 e. 972

15. Compute the area of the smallest region bounded by the graph of $y=|x|$ and $x^2+y^2=25$.

a. $\frac{25\pi}{4}$ b. $\frac{25\pi}{2}$ c. 25π d. $\frac{75\pi}{4}$ e. 5π

16. What is the remainder when 2018^{2018} is divided by 10?

a. 8 b. 4 c. 2 d. 6 e. 0

17. Let $A=\begin{bmatrix} x & 4 \\ 6 & y \end{bmatrix}$. If x and y are integers and $\det\left(A^2\right)=1$, which of the following is a possible value of $x+y$?

a. 14 b. 17 c. 20 d. 23 e. 26

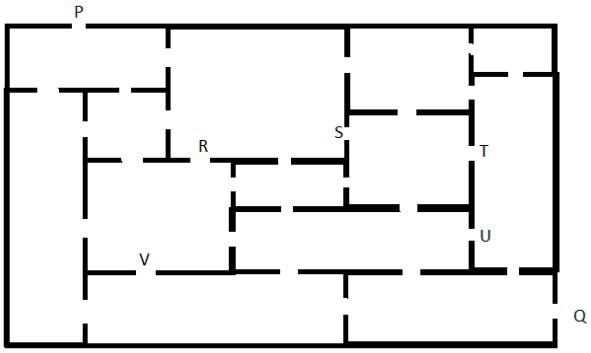
18. A fair six-sided die is rolled three times. What is the probability that the three numbers obtained are the lengths of the sides of an acute-angled triangle (i.e. one with all the angles less than 90°)?

a. $\frac{1}{6}$ b. $\frac{1}{3}$ c. $\frac{1}{4}$ d. $\frac{11}{36}$ e. $\frac{5}{18}$

19. A two-digit number not divisible by 9 is equal to k times the sum of its digits, where k is a positive integer. Which of the following is divisible by any possible value of k?

a. 210 b. 240 c. 280 d. 320 e. 350

20. A group of students visited a museum. They entered through doorway P and departed through doorway Q. In between, they passed through each doorway once and only once, except for one doorway. Which doorway did they not pass through?



a. R b. S c. T d. U e. V

SHORT ANSWER
Place the answer in the appropriate space.

66. What is the smallest possible positive angle in degrees that satisfies the equation $2\cos^2(x)+3\sin(x)=0$?

67. Around the circumference of a circle, mark 21 points, equally spaced, and label them 0, 1, 2, … , 20 in cyclic order. Color n of these points red so that no two pairs of red points are the same distance apart. What is the largest possible value of n?

68. How many different 4-tuples of nonnegative integers (a,b,c,d) satisfy the inequality $a+b+c+d\le 14$?

69. How many distinct solutions consisting of positive integers has the following system of linear equations?

$$\begin{aligned} x_1+x_2+x_3&=5\\ y_1+y_2+y_3&=5\\ z_1+z_2+z_3&=5\\ x_1+y_1+z_1&=5\\ x_2+y_2+z_2&=5\\ x_3+y_3+z_3&=5 \end{aligned}$$

70. Warren and Naida have a straight path, one meter wide and 16 meters long, that they need to pave. Warren brings home 16 paving stones, each 1 meter by 1 meter. Naida brings home 8 paving stones, each 1 meter by 2 meters. Assuming that they would consider using all square or all rectangular stones, or some of each (returning the unused stones for a refund), how many patterns are possible in paving the path?

Answer Key

1. E
2. B
3. D
4. C
5. A
6. D
7. A
8. C
9. B
10. E
11. D
12. E
13. B
14. A
15. A
16. B
17. E
18. D
19. C
20. C

66. 210
67. 5
68. 3060
69. 21
70. 1597